Year 12

Student Number

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2020 HSC Trial Examination

Year 12 Mathematics Extension 1

Examiner: AC

Reading time – 10 minutes

Writing time – 2 hours

General Instructions

- Write using black or blue pen.
- Read the instructions carefully – you are required to answer the questions in the space provided.
- If you use booklets, start each question in a separate writing booklet.
- Write your student name clearly on each page.
- Board-approved calculators may be used, unless stated otherwise.
- All diagrams must be drawn in pencil.
- Do not remove this question paper from the examination room.

Section	Guidance	Marks Available	Your Score
SECTION I	• *Type of Questions – Multiple Choice* • *Attempt Questions 1 - 10* • *Timing 15 minutes*	10	
SECTION II	• *Type of Questions – Multiple Choice* • *Attempt Questions 11 - 14* • *Timing 1hour 45 minutes*	60	
	Totals	70	

FINAL MARK	/ 70	%

Your examination paper begins overleaf.

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Section 1: 10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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1 Given $(x-2)$ is a factor of $x^3-8x^2+21x-A$, which of the following is the value of A?

(A) $A=-82$

(B) $A=-2$

(C) $A=2$

(D) $A=18$

2 Which of the following is the derivative of $\tan^{-1}(3x)$?

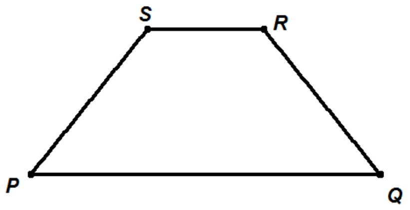
(A) $3\tan^{-1}x$

(B) $\dfrac{3}{1+x^2}$

(C) $\dfrac{3}{1+9x^2}$

(D) $3\sec^2 3x$

3 $PQRS$ is a trapezium where $\overrightarrow{PS}=\vec{p}$, $\overrightarrow{SR}=\vec{s}$ and $\overrightarrow{PQ}=2\overrightarrow{SR}$.



Which of the following is equivalent to $\overrightarrow{QS}$?

(A) $2\vec{s}+\vec{p}$

(B) $2\vec{s}-\vec{p}$

(C) $\vec{p}-2\vec{s}$

(D) $-\vec{p}-2\vec{s}$

4 Which of the following is the coefficient of x^4 in the expansion $\left(x+\frac{3}{x}\right)^8$?

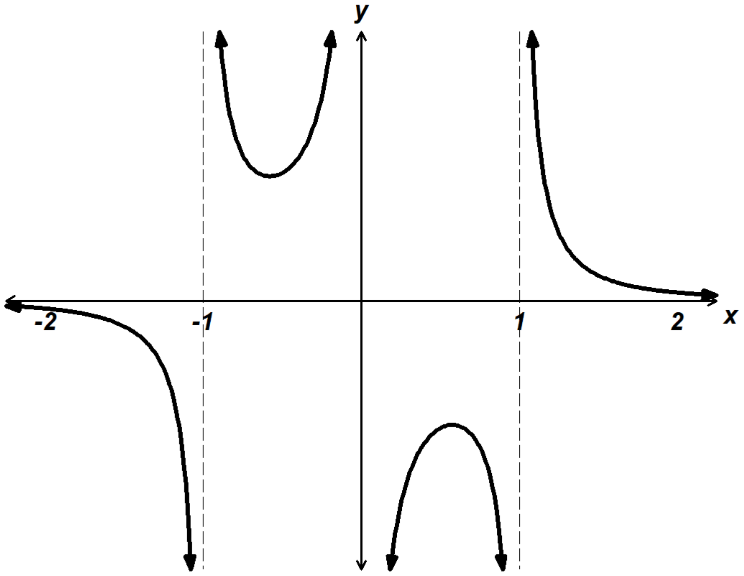
(A) 28

(B) 56

(C) 84

(D) 252

5



The graph above shows $y=\frac{1}{f(x)}$.

Which of the equations below best represents $f(x)$?

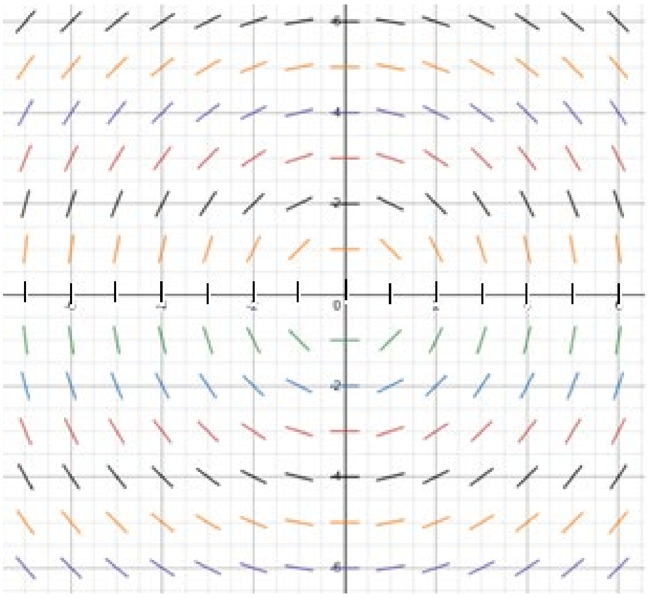
(A) $f(x)=x^2-1$

(B) $f(x)=x\left(x^2-1\right)$

(C) $f(x)=x^2\left(x^2-1\right)$

(D) $f(x)=x^2\left(x^2-1\right)^2$

6 The slope field for a first order differential equation is shown below.



Which of the following could be the differential equation represented?

(A) $\frac{dy}{dx} = \frac{x}{y}$

(B) $\frac{dy}{dx} = \frac{-x}{y}$

(C) $\frac{dy}{dx} = xy$

(D) $\frac{dy}{dx} = -xy$

7 Four female and four male students are to be seated around a circular table.
In how many ways can this be done if the males and females must alternate?

(A) $4! \times 4!$

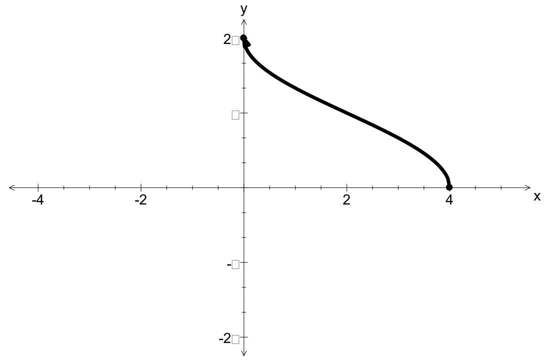
(B) $3! \times 4!$

(C) $3! \times 3!$

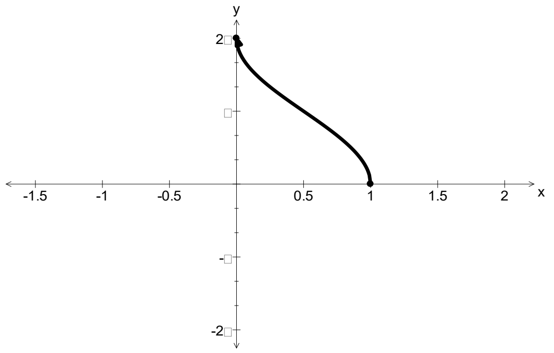
(D) $2 \times 3! \times 3!$

8 Which of the graphs below shows $y=2\cos^{-1}\left(\frac{x}{2}-1\right)$?

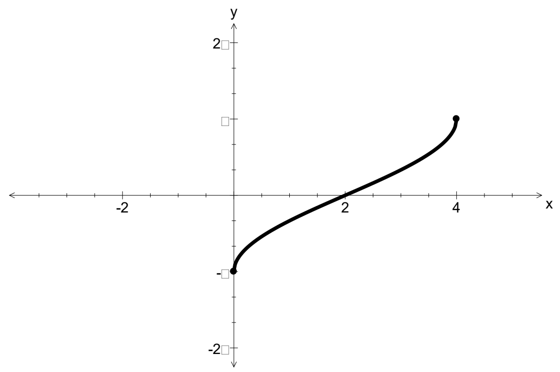
(A)



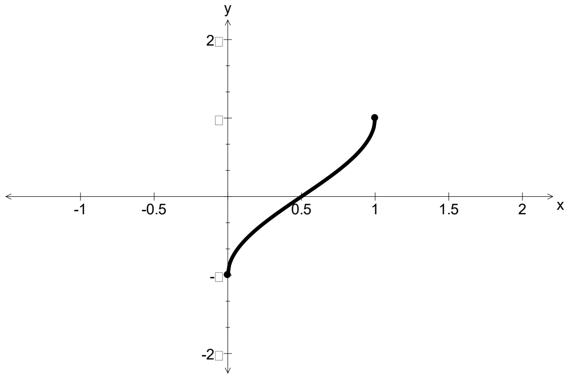
(B)



(C)



(D)



9 Which of the following expressions represents the area of the region bounded by the curve $y=\sin^3 x$ and the x-axis from $x=-\pi$ to $x=2\pi$? Use the substitution $u=\cos x$.

(A) $-\int_{-\pi}^{2\pi}(1-u^2)du$

(B) $-3\int_{0}^{\pi}(1-u^2)du$

(C) $-\int_{-1}^{1}(1-u^2)du$

(D) $3\int_{-1}^{1}(1-u^2)du$

10 Emma made an error proving that $2^n + (-1)^{n+1}$ is divisible by 3 for all integers $n \geq 1$ using mathematical induction. The proof is shown below.

<u>Step 1:</u> To prove $2^n + (-1)^{n+1}$ is divisible by 3 (*n* is an integer)

To prove true for $n = 1$

$2^1 + (-1)^{1+1} = 2 + 1$
$= 3 \times 1$ **Line 1**

Result is true for $n = 1$

<u>Step 2:</u> Assume true for $n = k$

ie. $2^k + (-1)^{k+1} = 3m$ (*m* is an integer) **Line 2**

<u>Step 3:</u> To prove true for $n = k + 1$

$$2^{k+1} + (-1)^{k+1+1} = 2(2^k) + (-1)^{k+2} \quad \textbf{Line 3}$$

$$= 2[3m + (-1)^{k+1}] + (-1)^{k+2} \quad \textbf{Line 4}$$

$$= 2 \times 3m + 2 \times (-1)^{k+2} + (-1)^{k+2}$$

$$= 3[2m + (-1)^{k+2}]$$

Which is a multiple of 3 since *m* and *k* are integers.

<u>Step 4:</u> True by induction

In which line did Emma make an error?

(A) Line 1

(B) Line 2

(C) Line 3

(D) Line 4

<u>Section II:</u> 60 marks

Attempt Questions 11 – 14

Allow about 1 hour and 45 minutes for this section

Answer each question in a separate writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) ***Start a new writing booklet***

(a) Consider the function $f(x) = x^2 - 4x + 6$.

(i) Explain why the domain of $f(x)$ must be restricted if $f(x)$ is to have an inverse function. **1**

(ii) Given that the domain of $f(x)$ is restricted to $x \leq 2$, find an expression for $f^{-1}(x)$. **2**

(iii) Given the restriction in part (ii), sketch $y = f^{-1}(x)$. **2**

(iv) The curve $y = f(x)$ with its restricted domain and the curve $y = f^{-1}(x)$ intersect at point *P*.

Find the coordinates of *P*. **1**

(b) Use the substitution $u = 1 + 2\tan x$ to evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{(1+2\tan x)^2 \cos^2 x}\,dx$. **3**

(Q11 continues on the next page)

(Q11 continued)

(c) Solve the equation $\cos x - \sin x = 1$, where $0 \le x \le 2\pi$. **3**

(d) The column (position) vector notation of 4 vectors is shown below.

$$P = \binom{-8}{-8} \quad Q = \binom{3}{6} \quad R = \binom{1}{5} \quad S = \binom{-5}{7}$$

Find the column (position) vector notation of:

(i) $\overrightarrow{PQ}$ **1**

(ii) $\overrightarrow{RS}$ **1**

(iii) $-\overrightarrow{PQ} - \overrightarrow{RS}$ **1**

End of Question 11. *Start a New Booklet.*

Question 12 (15 marks) ***Start a new writing booklet***

(a) A particle is moving in a straight line such that its displacement (x metres) from a fixed point O after t seconds is given by $x = \cos 2t + \sqrt{3}\sin 2t$.

(i) What is the maximum distance of the particle from O? **2**

(ii) When is the particle first at the origin? **1**

(b) A heated metal ball is dropped into a liquid. As the ball cools, its temperature, T °C, t minutes after it enters the liquid, is given by:

$$T = 400e^{-0\cdot 05t} + 25, \qquad t \geq 0$$

(i) Find the temperature of the ball as it enters the liquid. **1**

(ii) Find the value of t if $T = 300$. Answer correct to 3 significant figures. **1**

(iii) Find the rate at which the temperature of the ball is decreasing at the instant when $t = 50$. Give your answer in °C per minute to 3 significant figures. **2**

(iv) Using the equation for temperature T in terms of t, given above, to explain why the temperature of the ball can never fall to 20°C. **1**

(c) Find $\int_0^{\pi} \frac{4}{\sqrt{16-x^2}}\,dx$. **2**

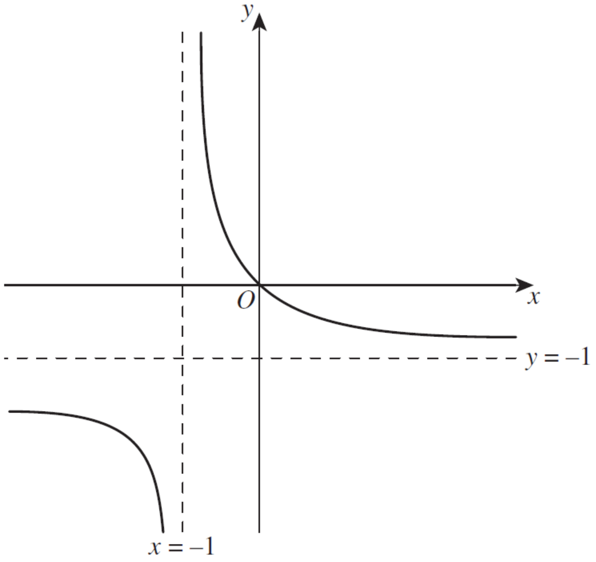
(d) (i) Use the substitution $t = \tan\frac{x}{2}$ to show that $\operatorname{cosec} x + \cot x = \cot\frac{x}{2}$. **2**

(ii) Hence evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\operatorname{cosec} x + \cot x)\,dx$. Answer in simplest exact form. **3**

End of Question 12. *Start a New Booklet.*

Question 13 (15 marks) ***Start a new writing booklet***

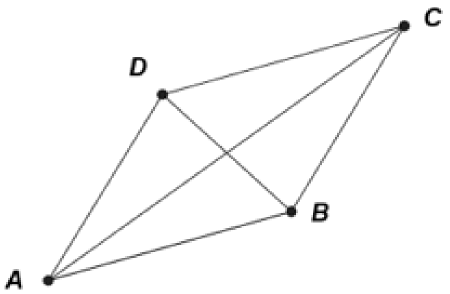
(a) The diagram below is the sketch of the graph of the function $f(x) = -\dfrac{x}{x+1}$.



(i) Sketch the graph of $y = (f(x))^2$, showing all asymptotes and intercepts. **2**

(ii) Solve the equation $(f(x))^2 = f(x)$. **1**

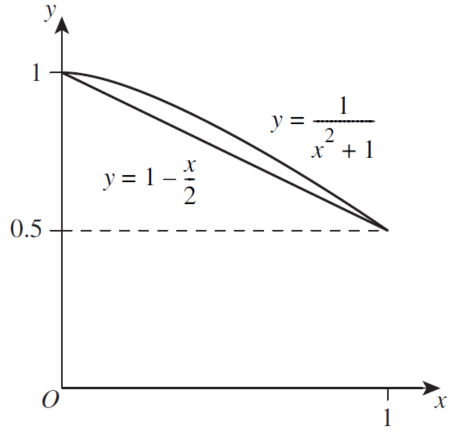
(b) *ABCD* is a rhombus with $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{d}$.



Use vector methods to prove that the diagonals of the rhombus are perpendicular to each other. **2**

(Q13 continues on the next page)

(c) The diagram shows the graph of $y = \dfrac{1}{x^2+1}$ and the graph of $y = 1 - \dfrac{x}{2}$ for $0 \le x \le 1$.



(i) Find the exact volume of the solid of revolution formed when the region bounded by the graph of $y = \dfrac{1}{x^2+1}$, the y-axis and the line $y = \dfrac{1}{2}$ is rotated about the y-axis. **2**

(ii) Find the exact volume of the solid of revolution formed when the region bounded by the graph of $y = 1 - \dfrac{x}{2}$, the y-axis and the line $y = \dfrac{1}{2}$ is rotated about the y-axis. **2**

(iii) Use the results from parts (c)(i) and (c)(ii) to show that $\dfrac{2}{3} < \ln 2$. **1**

(d) A multiple-choice test contains ten questions. Each question has four choices for the correct answer. Only one of the choices is correct.

(i) What is the probability of getting 70% correct with random guessing? **1**

(ii) What is the probability of getting at most 70% correct with random guessing? **2**

(e) A binomial random variable *X* has a mean of 15 and a variance of 10.
What are the parameters *n* and *p*? **2**

End of Question 13. *Start a New Booklet.*

Question 14 (15 marks) ***Start a new writing booklet***

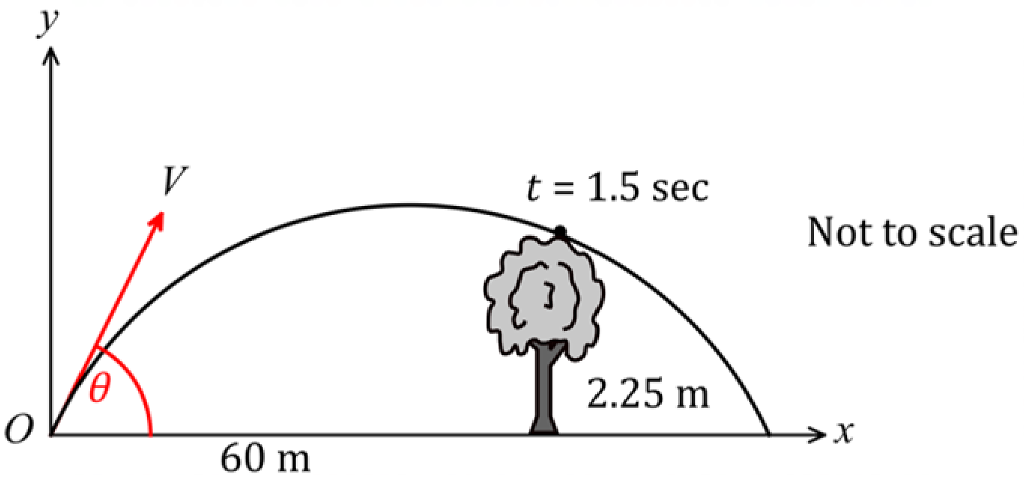
(a) Prove by mathematical induction that, for all integers $n \geq 1$,

$$\frac{2}{1 \times 3} + \frac{2}{2 \times 4} + \frac{2}{3 \times 5} + \ldots + \frac{2}{n(n+2)} = \frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}.$$ **3**

(b) A bag contains n red marbles and one blue marble. Three marbles are drawn (without replacement). The probability that the three marbles are red is $\frac{5}{8}$.

Find the value of n. **2**

(c) A golfer hits a golf ball from a point *0* with speed V ms^{-1} at an angle $\theta°$ above the horizontal, where $0 < \theta < \frac{\pi}{2}$. The ball just passes over a 2.25 m high tree after 1.5 seconds. The tree is 60 metres away from the point from which the ball was hit. Assume $g = 10$ ms^{-1}.



(i) What is the angle of projection of the golf ball to the nearest minute? Assume the horizontal and vertical displacements of the golf ball are given by the vector $\underset{\sim}{r}(t) = (Vt\cos\theta)i + (-5t^2 + Vt\sin\theta)j$. **3**

(ii) What is the initial speed (V ms^{-1}) of the golf ball, correct to the nearest whole number? **1**

(Q14 continues on the next page)

(d) The population, P, of animals in an environment in which there are scarce resources is increasing such that $\frac{dP}{dt} = P(100-P)$, where t is time.

When $t = 0, P = 10$.

(i) Show that $\frac{1}{100}\left(\frac{1}{P} + \frac{1}{100-P}\right) = \frac{1}{P(100-P)}$. **1**

(ii) Find an expression for P in terms of t. **3**

(e) The table shows selected values of a one-to-one differentiable function $g(x)$ and its derivative $g^{'}(x)$.

x	-1	0
$g(x)$	-5	-1
$g'(x)$	3	$\frac{1}{2}$

Let $f(x)$ be a function such that $f(x) = g^{-1}(x)$.

Find the value of $f'(-1)$. **2**

End of Question 14.

End of Examination.

NGS Trial Examination 2020
Year 12 Mathematics Extension 1
Worked solutions and marking guidelines

Section 1

10 marks

Question 1 (1 mark)

Solution	**Answer**	**Mark**
$(x-2)$ is a factor of $x^3-8x^2+21x-A$ $\therefore 8-32+42-A=0$ $A=18$	**D**	1

Question 2 (1 mark)

Solution	**Answer**	**Mark**
If $y=\tan^{-1}(3x)$ $\frac{dy}{dx}=\frac{1}{1+(3x)^2}\times 3$ $=\frac{3}{1+9x^2}$	**C**	**1**

Question 3 (1 mark)

Solution	**Answer**	**Mark**
$\overrightarrow{QS}=\overrightarrow{PS}-\overrightarrow{PQ}$ $=\vec{p}-2\vec{s}$	**C**	**1**

Question 4 (1 mark)

Solution	**Answer**	**Mark**
$T_{k+1}={}^8C_k(x)^{8-k}\left(\frac{3}{x}\right)^k$ $={}^8C_k x^{8-k}\left(3^k x^{-k}\right)$ $={}^8C_k(3)^k x^{8-2k}$ $x^4 \Rightarrow 8-2k=4$ $k=2$ the term is ${}^8C_2\times(3)^2=252$	**D**	**1**

Question 5 (1 mark)

Solution	**Answer**	**Mark**
By inspection and properties of $y=\frac{1}{f(x)}$	**B**	**1**

Question 6 (1 mark)

Solution	**Answer**	**Mark**
By considering slopes at different points of cartesian plane and testing with each differential equation	**B**	**1**

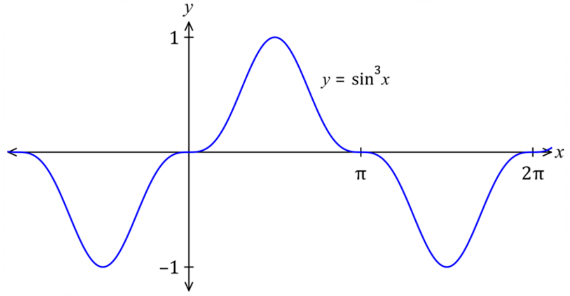
Question 7 (1 mark)

Solution	**Answer**	**Mark**
Seat females in $(4-1)!$ ways. Then seat males in $4!$ways $\therefore$ number of ways $=3!\times 4!$	**B**	**1**

Question 8 (1 mark)

Solution	**Answer**	**Mark**
For $y=2\cos^{-1}\left(\frac{x}{2}-1\right)$ Range $0\le y\le 2\pi$ Domain is $D:-1\le\frac{x}{2}-1\le 1$ $0\le x\le 4$	**A**	**1**

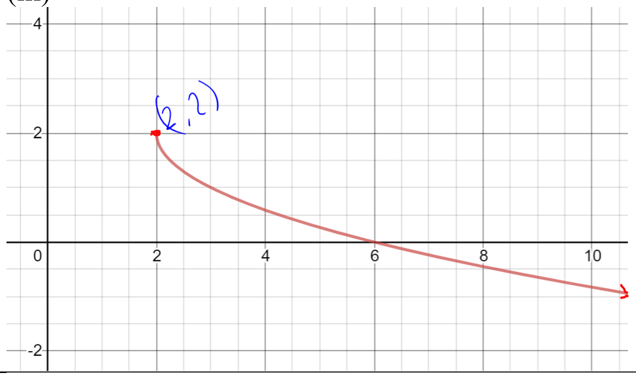
Question 9 (1 mark)

Solution	**Answer**	**Mark**
 There are 3 equivalent areas from $x=-\pi$ to $x=2\pi$ $u=\cos x$ $\frac{du}{dx}=-\sin x$ $du=-\sin x dx$ $x=0, u=1$ and $x=\pi, u=-1$ $A=3\times\int_0^{\pi}\sin^3 x dx$ $=-3\int_0^{\pi}(1-\cos^2 x)\times-\sin x\,dx$ $=-3\int_1^1(1-u^2)\,du$ $=3\int_{-1}^1(1-u^2)\,du$	**D**	**1**

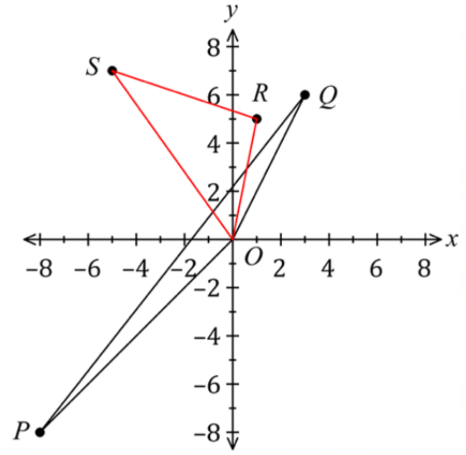
Question 10 (1 mark)

Solution	**Answer**	**Mark**
Step 3: To prove true for $n=k+1$ $2^{k+1}+(-1)^{k+1+1}=2(2^k)+(-1)^{k+2}$ $=2[3m-(-1)^{k+1}]+(-1)^{k+2}$ Error Line 4 $=2\times 3m-2\times(-1)^{k+1}-(-1)^{k+1}$ $=3[2m-(-1)^{k+1}]$ Which is a multiple of 3 since m and k are integers. Step 4: True by induction	**D**	**1**

Q11 (15 mark)

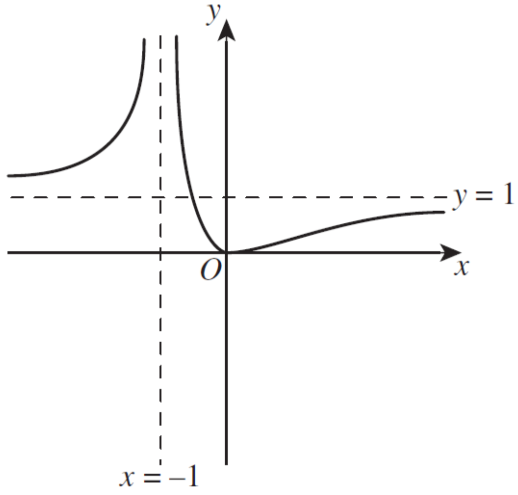
Solution	**Mark** *(Guide only)*
(a) (i) $f(x) = x^2 - 4x + 6$ is a parabola. Excluding the turning point at (2, 2), for each value of $f(x)$ in the range there are two x-values. Geometrically, this corresponds to a horizontal line intersecting the graph twice. If x and y are swapped, each x-value in the domain will have two y-values. Hence the inverse will not be a function.	1 Mark: Explains using the horizontal line test or equivalent merit.
(ii) Use the completing the square method to express $f(x)$ in turning point form: $f(x) = x^2 - 4x + 6$ $= (x-2)^2 + 2 \quad (x \le 2)$ Swap x and y, then make y the subject. $x = (y-2)^2 + 2$ $x - 2 = (y-2)^2$ $y - 2 = -\sqrt{x-2}$ ($\sqrt{x-2}$ is discarded as $y \le 2$) $y = -\sqrt{x-2} + 2$ $f^{-1}(x) = -\sqrt{x-2} + 2 \quad (x \ge 2)$	2 Marks: Correct Answer 1 Mark: Swaps x and y OR equivalent merit.
(iii) 	2 Marks: Correct shape and start at (2,2) 1 Mark: Correct shape OR starting point.
(iv) The curves $y = f(x)$ and $y = f^{-1}(x)$ have a common intersection with the line $y = x$. For example, attempting to solve $f(x) = x$ for x: $x^2 - 4x + 6 = x$ $x^2 - 5x + 6 = 0$ $x = 2, 3$ When $x = 2$, $y = 2$ and so $(2, 2)$ lies on the line $y = x$. When $x = 3$, $y = 1$ and so $(3, 1)$ does not lie on the line $y = x$. Therefore the coordinates of P are $(2, 2)$.	1 Mark: Correct solution

(b) Let $u = 1 + 2\tan x$. $\frac{du}{dx} = 2\sec^2 x = \frac{2}{\cos^2 x} \Rightarrow dx = \frac{\cos^2 x}{2}du$ When $x = 0$, $u = 1$ and when $x = \frac{\pi}{4}$, $u = 3$. $\int_0^{\frac{\pi}{4}} \frac{1}{(1+2\tan x)^2\cos^2 x}dx = \int_1^3 \frac{1}{2u^2}du$ $= -\left[\frac{1}{2u}\right]_1^3$ $= -\left(\frac{1}{6} - \frac{1}{2}\right)$ $= \frac{1}{3}$	3 Marks: Correct solution 2 Marks: Finds expression for integral in terms of u, or equivalent merit. 1 Mark: Derives u=1+2tanx correctly
(c) Substituting $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$ where $t = \tan\frac{1}{2}x$ into $\cos x - \sin x = 1$ and expressing $1 = \frac{1+t^2}{1+t^2}$ gives: $\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} = \frac{1+t^2}{1+t^2}$ $\frac{1-t^2-2t-1-t^2}{1+t^2} = 0$ $\frac{-2(t^2+t)}{1+t^2} = 0$ $t^2 + t = 0$ $t(t+1) = 0$ $t = -1, 0$ $\tan\frac{1}{2}x = -1, 0$ $\tan\frac{1}{2}x = 0 \Rightarrow \frac{1}{2}x = 0, \pi$ $\tan\frac{1}{2}x = -1$ tan is negative in the second quadrant and the related angle is $\frac{\pi}{4}$. $\tan\frac{1}{2}x = -1 \Rightarrow \frac{x}{2} = \frac{3\pi}{4}$ So $x = 0, \frac{3\pi}{2}, 2\pi$.	3 Marks: Correct solution 2 Marks: Determines that $\tan(\frac{1}{2}x) = -1, 0$ 1 Mark: Attempts to form a quadratic equation in *t* with some correct working OR equivalent merit.

(d) (i)  $\overrightarrow{PQ} = \begin{pmatrix} 11 \\ 14 \end{pmatrix}$	1 Mark: Correct answer.
(ii) $\overrightarrow{RS} = \begin{pmatrix} -6 \\ 2 \end{pmatrix}$	1 Mark: Correct answer.
(iii) $-\overrightarrow{PQ} - \overrightarrow{RS} = \begin{pmatrix} -5 \\ -16 \end{pmatrix}$	1 Mark: Correct answer.

Q12 (15 mark)	
(a) (i) Using auxiliary angle: $x = \cos 2t + \sqrt{3}\sin 2t = 2\cos\left(2t - \frac{\pi}{3}\right)$ $\therefore$ maximum distance from $O = 2$ metres	2 marks: Correct Answer 1 Mark: Some attempt at Auxiliary Angle with some correct working OR equivalent merit
(ii) $2\cos\left(2t - \frac{\pi}{3}\right) = 0$ $2t - \frac{\pi}{3} = \frac{\pi}{2}$ $t = \frac{5\pi}{12}$ seconds	1 Mark: Correct answer.
(b) (i) Ball enters the liquid when $t = 0$ $T = 400e^{-0\cdot 05t} + 25$ $= 400e^{-0\cdot 05\times 0} + 25 = 425\,°\text{C}$	1 Mark: Correct answer.
(ii) $300 = 400e^{-0\cdot 05t} + 25$ $e^{-0\cdot 05t} = \frac{275}{400}$ $-0.05t = \ln\left(\frac{11}{16}\right)$ $t = 7.4938\ldots \approx 7.49$ min (3 sig. fig.)	1 Mark: Correct answer.
(iii) $T = 400e^{-0\cdot 05t} + 25$ $\frac{dT}{dt} = -20e^{-0\cdot 05t}$ $= -20e^{-0\cdot 05\times 50}$ $= -1.6416\ldots \approx -1.64°\text{C/min}$ $\therefore$ Rate of decrease is 1.64°C per minute.	2 Marks: Correct answer. 1.5 marks: Only giving -1.64/min 1 Mark: Differentiates correctly to find the rate of change.
(iv) When t approaches infinity then $e^{-0\cdot 05t} \to 0$ $\therefore T > 25$ and can never fall to 20°C.	1 Mark: Correct answer. **(Must <u>use the equation</u> to show correctly)**
(c)	

$\int_0^{\pi} \frac{4}{\sqrt{16-x^2}}\,dx = 4\left[\sin^{-1}\left(\frac{x}{4}\right)\right]_0^{\pi}$ $= 4\left[\sin^{-1}\left(\frac{\pi}{4}\right) - \sin^{-1}(0)\right]$ $= 4\left[\frac{1}{\sqrt{2}} - 0\right]$ $= \frac{4}{\sqrt{2}} = 2\sqrt{2}$	2 Marks: Correct answer. 1 Mark: Finds the correct integration.
(d) (i) $\text{LHS} = \text{cosec}x + \cot x$ $= \frac{1+t^2}{2t} + \frac{1-t^2}{2t}$ $= \frac{1+t^2+1-t^2}{2t}$ $= \frac{1}{t}$ $= \cot\frac{x}{2}$ $= \text{RHS}$	2 Marks: Correct answer. 1 Mark: Writes cosec$x$ and cot$x$ in terms of t.
(ii) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\text{cosec}x + \cot x)dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cot\frac{x}{2}\,dx$ $= 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{0.5\cos\frac{x}{2}}{\sin\frac{x}{2}}\,dx$ $= 2\left[\ln\left(\sin\frac{x}{2}\right)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$ $= 2\left[\ln\left(\sin\frac{\pi}{4}\right) - \ln\left(\sin\frac{\pi}{6}\right)\right]$ $= 2\left(\ln\frac{1}{\sqrt{2}} - \ln\frac{1}{2}\right)\vert$ $= 2\ln\left(2^{-\frac{1}{2}} \div 2^{-1}\right)$ $= 2\ln 2^{\frac{1}{2}}$ $= \ln 2$	3 Marks: Correct answer. 2 Marks: Makes significant progress. OR uses logs correctly 1 Mark: Finds the primitive function.

Q13 (15 mark)	
(a) (i)  	2 Marks: Sketches correct graph with asymptotes at x=-1 and y=1 1 Mark: Shows min turning point at origin OR equivalent merit.
(ii) $(f(x))^2 = f(x) \Rightarrow f(x)(f(x) - 1) = 0$ So $f(x) = 1$ or $f(x) = 0$. $-\frac{x}{x+1} = 1 \Rightarrow x = -\frac{1}{2}$ Hence $x = -\frac{1}{2}$ or $x = 0$. **OR** The graphs of $y = f(x)$ and $y = (f(x))^2$ intersect at O, where $x = 0$. The graphs of $y = f(x)$ and $y = (f(x))^2$ intersect on the line $y = 1$, where $x = -\frac{1}{2}$.	1 Mark: Correct solution
(b) $\overrightarrow{AC} = \mathbf{a} + \mathbf{d}, \overrightarrow{DB} = \mathbf{a} - \mathbf{d}$ $\overrightarrow{AC} \bullet \overrightarrow{DB} = (\mathbf{a} + \mathbf{d}) \bullet (\mathbf{a} - \mathbf{d})$ $= \mathbf{a} \bullet \mathbf{a} - \mathbf{d} \bullet \mathbf{d}$ $= \lvert\mathbf{a}\rvert^2 - \lvert\mathbf{d}\rvert^2$ $= 0$ (sides of rhombus have equal length) But $\overrightarrow{AC} \bullet \overrightarrow{DB} = \lvert\mathbf{a}\rvert \times \lvert\mathbf{d}\rvert \cos\theta$ (where θ is the angle between the vectors) $\therefore \cos\theta = 0$ $\theta = 90^\circ$ *i.e.* diagonals are perpendicular	2 marks: Correct Answer. 1 mark: Writes $\overrightarrow{AC} = \boldsymbol{a} + \boldsymbol{d}$
(c)	2 marks: Correct Answer. 1 mark: Gives correct integral for volume of revolution

(i) Rearranging $y=\frac{1}{x^2+1}$ to express x^2 in terms of y gives $x^2=\frac{1}{y}-1$. $V=\pi\int_{\frac{1}{2}}^{1}\left(\frac{1}{y}-1\right)dy$ $=\pi\left[\ln\lvert y\rvert-y\right]_{\frac{1}{2}}^{1}$ $=\pi\left(\ln 1-1-\left(\ln\frac{1}{2}-\frac{1}{2}\right)\right)$ $=\pi\left(\ln 2-\frac{1}{2}\right)$	
(ii) Rearranging $y=1-\frac{x}{2}$ to express x in terms of y gives $x=2(1-y)$. $V=\pi\int_{\frac{1}{2}}^{1}(4(1-y)^2)dy$ $=-\frac{4\pi}{3}\left[(1-y)^3\right]_{\frac{1}{2}}^{1}$ $=-\frac{4\pi}{3}\left(0-\frac{1}{8}\right)$ $=\frac{\pi}{6}$ Alternatively: The solid formed is a cone of radius 1 and height $\frac{1}{2}$. Substituting these values into $V=\frac{1}{3}\pi r^2h$ gives: $V=\frac{1}{3}\times\pi\times 1^2\times\frac{1}{2}$ $=\frac{\pi}{6}$	2 marks: Correct Answer. 1 mark: Gives correct integral for volume of revolution
(iii) From the diagram, it can be reasoned that $\pi\left(\ln 2-\frac{1}{2}\right)>\frac{\pi}{6}$. So $\ln 2-\frac{1}{2}>\frac{1}{6}\Rightarrow\ln 2>\frac{2}{3}$.	1 Mark: Correct answer.
(d) (i) Let p be the probability of getting the correct answer. $p=\frac{1}{4}, n=10$ $P(X=x)={}^{10}C_x\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{10-x}$ $P(X=7)={}^{10}C_7\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^{10-7}$ $=\frac{405}{131072}$	1 Mark: Correct answer.

(ii) $P(X \le 7) = 1 - \left(P(8) + P(9) + P(10)\right)$ $= 1 - \left({}^{10}C_8 \left(\frac{1}{4}\right)^8 \left(\frac{3}{4}\right)^{10-8} + {}^{10}C_9 \left(\frac{1}{4}\right)^9 \left(\frac{3}{4}\right)^{10-9} + {}^{10}C_{10} \left(\frac{1}{4}\right)^{10} \left(\frac{3}{4}\right)^{10-10} \right)$ $= 0.99958...$	2 Marks: Correct answer. 1 Mark: Uses the complementary event or shows some understanding.
(e) $E(X) = np = 15$ ① $Var(X) = np(1-p) = 10$ ② Substituting equation ① into ② $15 \times (1-p) = 10$ $1 - p = \frac{10}{15} = \frac{2}{3}$ or $p = \frac{1}{3}$ Substituting $p = \frac{1}{3}$ into equation ① $n \times \frac{1}{3} = 15$ $n = 45$ $\therefore$ Parameters are $n = 45$ and $p = \frac{1}{3}$	2 Marks: Correct answer. 1 Mark: Finds one of the parameters or shows some understanding.

Q14 (15 mark)	
(a) Consider $n = 1$. $\text{LHS} = \frac{2}{1 \times 3} = \frac{2}{3}$ and $\text{RHS} = \frac{3}{2} - \frac{2(1)+3}{(1+1)(1+2)} = \frac{4}{6} = \frac{2}{3} = \text{LHS}.$ The statement is true when $n = 1$. Suppose true for $n = k$. So $\frac{2}{1 \times 3} + \frac{2}{2 \times 4} + \frac{2}{3 \times 5} + \ldots + \frac{2}{k(k+2)} = \frac{3}{2} - \frac{2k+3}{(k+1)(k+2)}.$ Show it is true for $n = k + 1$; that is, $\frac{2}{1 \times 3} + \frac{2}{2 \times 4} + \frac{2}{3 \times 5} + \ldots + \frac{2}{k(k+2)} + \frac{2}{(k+1)(k+3)} =$ $\frac{3}{2} - \frac{2(k+1)+3}{((k+1)+1)((k+1)+2)}$ $\text{LHS} = \frac{2}{1 \times 3} + \frac{2}{2 \times 4} + \frac{2}{3 \times 5} + \ldots + \frac{2}{k(k+2)} + \frac{2}{(k+1)(k+3)}$ $= \frac{3}{2} - \frac{2k+3}{(k+1)(k+2)} + \frac{2}{(k+1)(k+3)}$ $= \frac{3}{2} - \frac{(2k+3)(k+3) - 2(k+2)}{(k+1)(k+2)(k+3)}$ $= \frac{3}{2} - \frac{2k^2 + 7k + 5}{(k+1)(k+2)(k+3)}$ $= \frac{3}{2} - \frac{(2k+5)(k+1)}{(k+1)(k+2)(k+3)}$ $= \frac{3}{2} - \frac{2k+5}{(k+2)(k+3)}$ $= \frac{3}{2} - \frac{2(k+1)+3}{((k+1)+1)((k+1)+2)}$ $= \text{RHS}$ If true for $n = k$, then true for $n = k + 1$. Hence, by mathematical induction, true for $n \geq 1$.	3 Marks: Correct proof 2 Marks: Establishes the induictive step OR equivalent merit. 1 Mark: Establishes the n=1 case or equivalent merit.
(b) $\text{Prob}(3 \text{ red marbles drawn}) = \frac{{}^{n}C_3}{{}^{n+1}C_3}$ ⇩ $\frac{n}{n+1} \times \frac{n-1}{n} \times \frac{n-2}{n-1} = \frac{5}{8}$ $\frac{n-2}{n+1} = \frac{5}{8}$ $n = 7$ ⇩ $\frac{{}^{n}C_3}{{}^{n+1}C_3} = \frac{5}{8}$ $\frac{\left(\frac{n(n-1)(n-2)}{3 \times 2 \times 1}\right)}{\frac{(n+1)(n)(n-1)}{3 \times 2 \times 1}} = \frac{5}{8}$ $\frac{n-2}{n+1} = \frac{5}{8}$ $8n - 16 = 5n + 5$ $n = 7$	2 marks: Correct Answer 1 mark: Writes correct answer for probability

(c) (i) After 1.5 seconds $x = 60$ and $y = 2.25$ $60 = 1.5V\cos\theta$ $V\cos\theta = 40$ ① $2.25 = -5 \times 1.5^2 + 1.5V\sin\theta$ $13.5 = 1.5V\sin\theta$ $V\sin\theta = 9$ ② Dividing the two equations $\frac{V\sin\theta}{V\cos\theta} = \frac{9}{40}$ $\tan\theta = \frac{9}{40}$ $\theta = \tan^{-1}\frac{9}{40} = 12°41'$ ∴ Golf ball has an angle of projection of 12˚41'.	3 marks: Correct answer. 2 marks: Makes significant progress. 1 mark: Sets up the two equations or shows some understanding.
(ii) Using equations ① and ② $(V\sin\theta)^2 + (V\cos\theta)^2 = 9^2 + 40^2$ $V^2(\sin^2\theta + \cos^2\theta) = 9^2 + 40^2$ $V = \sqrt{9^2 + 40^2}$ $V = 41\ \text{ms}^{-1}$ ∴ Speed of the gold ball is 41 ms^{-1}	1 Mark: Correct answer.
(d) (i) $\frac{1}{100}\left(\frac{1}{P} + \frac{1}{100-P}\right) = \frac{1}{100}\left(\frac{100-P+P}{P(100-P)}\right) = \frac{1}{P(100-P)}$	1 mark: Correctly shows the result

(ii) $\frac{dP}{dt}=P\left(100-P\right)$ $\frac{1}{P\left(100-P\right)}dP=dt$ $\int\frac{1}{P\left(100-P\right)}dP=\int dt$ $\int\frac{1}{100}\left(\frac{1}{P}+\frac{1}{100-P}\right)dP=\int dt$ $\int\left(\frac{1}{P}+\frac{1}{100-P}\right)dP=100\int dt$ $\log_e P-\log_e\left(100-P\right)=100t+c$ $t=0, P=10 \Rightarrow \log_e 10-\log_e 90=c$ $c=\log_e\frac{1}{9}$ $\log_e\left(\frac{P}{100-P}\right)=100t+\log_e\frac{1}{9}$ $\left(\frac{P}{100-P}\right)=e^{100t+\log_e\frac{1}{9}}$ $\frac{P}{100-P}=\frac{1}{9}e^{100t}$ $\Longrightarrow$ $\frac{100-P}{P}=9e^{-100t}$ $9P=100e^{100t}-Pe^{100t}$ $9P+Pe^{100t}=100e^{100t}$ $P=\frac{100e^{100t}}{9+e^{100t}}$ $\frac{100}{P}-1=9e^{-100t}$ $\frac{100}{P}=1+9e^{-100t}$ $P=\frac{100}{1+9e^{-100t}}$	3 marks: Correct answer 2 marks: Integrates correctly without finding the constant OR Integrates correctly, finds c, but leaves as t = f(P). 1 mark: Makes some progress. Ie. seperates the differential equation and attempts to integrate.
(e) From the table, $f(x)=g^{-1}(x)$ and so $f(-1)=g^{-1}(-1)=0$. $f'(-1)=\frac{1}{g'(f(-1))}$ $=\frac{1}{g'(0)}$ $=\frac{1}{\frac{1}{2}}$ $=2$	3 marks: Gives the correct solution 2 marks: Determines $f(-1)=g^{-1}(-1)=0$ AND $f'(-1)=\frac{1}{g'(f(-1))}$ . . . . 1 mark: Determines $f(-1)=g^{-1}(-1)=0$ OR $f'(-1)=\frac{1}{g'(f(-1))}$ . . . .